Chapter 2

Theory of Financial Laws

2.1. Indifference relations and exchange laws for simple financial operations

Let us consider again the indifference relation, indicated by ≈ in (1.1), which depends on the judgment of an economic operator which gives rise to indifferent supplies with the process described in section 1.2.

In a *loan operation* of the amount S at time T the economic operator can calculate the repayment value S' in $T' > T$ such that $(T',S') \approx (T,S)$. Therefore, $S' \geq S$ is calculated according to a function (subjective) of S, T, T' and it is written as

$$S' = f_c\,(S, T; T') \tag{2.1}$$

where f_c is the *accumulation function* (given that in S' the repayment of S and the incorporation of the possible interest is included) that realizes indifference.

In a *discounting operation*, at time $T'' < T'$, of amount S' with maturity T', let $S'' \leq S'$ be the discounted value so that subjectively $(T'',S'') \approx (T',S')$. We then have

$$S'' = f_a\,(S', T'; T'') \tag{2.2}$$

where f_a is the *discounting function* (because S' is discounted at time T'' with a possible reduction due to anticipation of availability) that realizes indifference.

It is obvious that if two operators, one at each side of a loan or discounting contract, want to realize an advantageous contract according to their preference scale, it is not always possible for them to do so.

It can be the case that, in a loan in T' of the principal S', indicating by S''_a the indifferent accumulated amount (= min acceptable) for the *lender* to cash in T'' and by S''_b the indifferent accumulated amount (= max acceptable) for the *borrower* to pay out in T'', if $S''_b < S''_a$ the contract is not stipulated. In the same way, we can prove that, in a discounting operation of the capital S' at maturity T', indicating by S''_a the present indifferent value (= max acceptable) for the *lender* to pay out in $T'' < T'$ and by S''_b the present indifferent value (= min acceptable) for the *borrower* to cash in $T'' < T'$, if $S''_a < S''_b$ the contract is not stipulated.

EXAMPLE 2.1.– Let us suppose that Mr. Robert, who is lending the amount S' at time T' for the period (T',T''), wants to cash in T'' at least $1.09 \cdot S'$. At the same time Mr. George, who is borrowing S' for the same time interval, wants to pay back in T'' no more than $1.07 \cdot S'$. It is obvious that in this way they will not proceed with the loan contract. Indeed:

– with $S'' < 1.07 \cdot S'$, the lender prefers not to lend;

– with $1.07 \cdot S' < S'' < 1.09 \cdot S'$, the lender prefers not to lend and the borrower prefers not to borrow;

– with $S'' > 1.09 \cdot S'$, the borrower prefers not to borrow.

EXAMPLE 2.2.– Let us suppose that Mr. John wants to discount a bill from Mr. Tom, which is amount S' for the time from T' to $T''<T'$ offering a discounted value not greater than $0.92 \cdot S'$, while Mr. Tom wants to offer this discount for an amount not lower than $0.94 \cdot S'$. It is clear that the contract cannot be reached, because each discounted amount is considered disadvantageous by at least one of the parties.

To further consider the economic theory of market prices, we carry on our analysis using *objective logic* and supposing that the operators, in a specific market, want a fair contract between two supplies (T,S) and (T',S') in a loan, if their fundamental quantities satisfy equation (2.1); and in the same way, for a discount, which is a type of loan, if equation (2.2) is satisfied. We will now talk about a *fair contract* if equation (2.1) or equation (2.2) is satisfied, but as *favorable* (or *unfavorable*) for one of the parties if the equations are not satisfied. Trade contracts between two supplies (T',S') and (T'',S'') give rise to *simple financial operations.* As already mentioned in Chapter 1:

– if $T'' > T'$ (= *loan* or *investment*), the parties consider fair the interest $S''-S'$ as the payment for the lending of S' from T' to T'', as delayed payment in T''; then S'' is called *accumulated amount* in T'' of the amount S' lent in T';

– if $T'' < T'$ (= *discount* or *anticipation*), both parties consider fair the interest S'-S'' for the discount of S' from T' to T'', as advance payment in T''; then S'' is called *discounted value* from time T'' of the amount S' to maturity T'[1].

The indifference relation thus assumes a collective value. The function f_c defined in equation (2.1) is an *accumulation law* (or *interest law),* while the function f_a defined in equation (2.2) is a *discount law*. Referring now to the case of positive interest and fixing S and T in equation (2.1), the value S' is an increasing function of T'; fixing S' and T' in equation (2.2), and the value S'' is also an increasing function of T'', because it decreases when T'' decreases.

Applying equation (2.1) and then equation (2.2) with $T'' = T$, we obtain the present value in T of the accumulated amount in T' of S invested in $T \leq T'$, given by

$$S^* = f_a\,[\{ f_c\,(S,T;T')\},T';T\,] \tag{2.3}$$

If $\forall(S,T,T')$ is $S^* = S$, the f_a neutralizes the effect of f_c, acting as the inverse function, and the following investment or anticipation operation is called the *corresponding operation*; in this case the laws expressed by f_c and f_a are said to be *conjugated*.

Unifying the cases $T \leq T'$ and $T>T'$, we can talk of an *exchange law* given by a function f that gives the amount S' payable in T' and exchangeable[2] with S payable in T. It follows that

$$S' = f\,(S,T;T') \tag{2.4}$$

where if $T \leq T'$ then $f = f_c$, whereas if $T > T'$ then $f = f_a$.

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1 Lending and discounting operations are the same thing because in both cases there is an exchange of a lower amount in a previous time for a greater amount in a future time. The only difference is that in the first case the lower and previous amount is fixed, whereas in the second case the greater and future amount is fixed.

2 We will not use “equivalent” – even if it is used in practice – in the cases that we will consider later where ≈ gives rise to an equivalence relation (see footnote 6 of Chapter 1).

Let us consider some properties of the indifference relation $\approx$:

1) reflexive property

If $\forall (T,S)$ we have $(T,S) \approx (T,S)$, we will say that $\approx$ satisfies the reflexive property[3];

2) symmetric property

If $\forall (S,T,T')$, from $(T,S) \approx (T',S')$ follows $(T',S') \approx (T,S)$, we will say that $\approx$ satisfies the symmetric property[4];

3) property of proportional amounts

If $\forall (S,T,T')$, $\forall k>0$, from $(T,S) \approx (T',S')$ follows $(T,kS) \approx (T',kS')$, we will say that $\approx$ satisfies the property of proportional amounts.

Because of criteria c) and d), if T′-T the amount in T′ exchangeable with S in T is the same as S. Therefore in the set $\mathcal{P}$ of financial supplies the relation $\approx$ always satisfies the reflexive law. We can then define the exchange law for all three variables as

$$f(S,T;T') = \begin{cases} f_c(S,T;T'), & if\ T < T' \\ S, & if\ T = T' \\ f_a(S,T;T'), & if\ T > T' \end{cases} \tag{2.5}$$

If the symmetric law holds in the considered set P, then

$$S = f_a[\{ f_c (S,T,T')\},T',T],\ \forall (S,T,T'),\ T<T'\ ^{5} \tag{2.6}$$

In this case, recalling (2.3), the laws f_c and f_a are conjugated, and because of (2.4), (2.5) can be written in the form

$$S = f\,[\{ f\,(S,T,T')\},T',T],\ \forall (S,T,T') \tag{2.6'}$$

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3 Let us recall that a binary relation $\mathcal{R}$ between elements a, b, ... of a set $\mathcal{H}$ satisfies the reflexive law if: $a\mathcal{R}a$, $\forall a \in \mathcal{H}$.

4 Let us recall that a binary relation $\mathcal{R}$ between elements a, b, ... of a set $\mathcal{H}$ satisfies the symmetric law if: $a\mathcal{R}b \Rightarrow b\mathcal{R}a$, $\forall a,b \in \mathcal{H}$.

5 If $T>T'$ is given, f_c and f_a have to be exchanged in (2.6).

which remains valid with the same f if the primed values are changed with the unprimed values and vice versa[6].

If, in the considered set $\mathcal{P}$, the property of proportional amounts holds, f as defined in (2.4) is *linear homogenous* compared to the amount[7].

2.2. Two variable laws and exchange factors

Let us continue the analysis of exchange laws *the reflexive* and *proportional amount properties* assumed to be valid for $\approx$. Due to the second property, it is possible to transform (2.1) in the multiplicative form

$$S' = S \cdot m(T,T'),\ T \leq T' \tag{2.1'}$$

where $m(T,T')$, increasing with respect to T', is called the *accumulation factor* and expresses the *accumulation law* only as a function of the two temporal variables; in the same way it is possible to transform (2.2) in the form

$$S'' = S' \cdot a(T',T''),\ T' \geq T'' \tag{2.2'}$$

where $a(T',T'')$, increasing with respect to T'', is called the *discounted factor* and expresses the *discounting law* only as a function of the two temporal variables. We will now address the *two variables laws.*

The *reflexive law* for $\approx$ is now equivalent to

$$m(T,T) = a(T,T) = 1,\ \forall T \tag{2.7}$$

Furthermore if, using $T'' = T$ in systems (2.1') and (2.2'), we obtain $S'' = S$, i.e. the symmetric property is valid for $\approx$, the laws $m(\cdot)$ and $a(\cdot)$ satisfy

$$m(T,T') \cdot a(T',T) = 1,\ \forall T \leq T' \tag{2.8}$$

6 The symmetric case – far from being realistic in the contracts with companies and banks, due to the different conditions and onerousness of the lending market (which leads to costs for the companies) compared to the investment market (which leads to profits for the companies) – can be applied to the contracts between persons or linked companies and, from a theoretical point of view, makes it possible to deal with the two systems in a similar manner.

7 The property of proportional amounts is normally used in theoretical schemes, but should only be used with smaller amounts. The financial profits for the unit of invested capital can change according to the value of the capital and the contractual strength of the investors.

Equation (2.8) shows that *conjugated laws for the same time interval give rise to reciprocal factors.*

When describing (2.4) in detail, we consider *the exchange law of two variables* characterized by *the exchange factor z(X,Y)*, a pure number increasing with respect to *Y*, defined using

$$z(X,Y) = \begin{cases} m(X,Y)\,, & if\ X < Y \\ 1 \quad , & if\ \ X = Y \\ a(X,Y)\,, & if\ X > Y \end{cases} \tag{2.5'}$$

(2.5′) being a particular case of (2.5).

To summarize, given an indifference relation ≈, the corresponding exchange law expressed by the factor $z(X,Y)$, such that $(X,S_1) \approx (Y,S_2)$, is equivalent to $S_2 = S_1\ z(X,Y)$. The exchange factor $z(X,Y)$ is a function defined for each couple (X,Y) of exchange times, which “brings” the values from X to Y forward (= accumulation) if $X<Y$ and backward (= discounting) if $X>Y$.

We will now assume that

$$z(X,Y) > 0,\ \forall (X,Y) \tag{2.5''}$$

(considering, if needed, only the part of the definition set for the function *z* where such a condition holds) in order that it cannot be possible that an encashment (payment) can never be indifferent to a payment (encashment) with different time maturity.

In geometric terms, let us consider the Cartesian plane *OXY* with the points $G \equiv (X,Y)$ with the aforementioned meaning[8]. The exchange factor is then the point function $z(G)$. Because of (2.5'), $z(G)=1$ if *G* is *on* the bisector of the coordinate axes. Furthermore, if *G* is *over* the bisector (i.e. if $X<Y$), then $z(G) = m(X,Y) > 1$; otherwise, if *G* is *under* the bisector (i.e. if $X>Y$), $z(G) = a(X,Y)<1$ and more precisely because of (2.5"): $0<a(X,Y)<1$[9].

8 Note that the functions $m(X,Y)$ and $a(X,Y)$ are defined in the disjoint half-planes $X<Y$ and $X>Y$, i.e. over and under the bisector of coordinate axes. It can be useful to extend their definition on the bisector $Y=X$, recalling (2.7) and putting $m(X,X) = a(X,X) = 1$.

9 (2.5') brings to a general formulation of exchange value of two variables, which does not imply the symmetry of financial relations. It follows that the law $z(X,Y)$ can be used to schematize not just the time variability of the cost and profit parameters, but also their difference in investment and discount operations which are of interest to any company. For example, if a company obtains liquid assets through anticipation of future credits and uses

Recalling the considerations of Chapter 1 (especially criteria d) for positive amounts, given that $z(X,Y)$ is the exchange value of unitary amount), in the hypothesis of positive returns for the money the contour curves $z(X,Y)$ = const. are graphs of strictly increasing functions $Y=\psi(X)$[10].

If relation ≈ expressed by $z(X,Y)$ satisfies *the symmetric property*, as a particular case of (2.6') the below condition follows:

$$z(X,Y)\cdot z(Y,X) = 1;\ \forall(X,Y) \tag{2.9}$$

If $z(X,Y)$ satisfies (2.9), then it defines a couple of two-variable financial interest and discount laws which are conjugated.

It is obvious that if the indifference relation is symmetric, it is enough to be able to define $z(X,Y)$ in one of the two half-planes to obtain the value of z in the second half-plane using the following rule: *the values of z for points which are symmetric with respect to the bisector are reciprocal*. In this case $z(X,Y) = 1/z(Y,X)$, $\forall(X,Y)$ then the couples of contour curves of accumulation factor $z = k > 1$ and discount factor $z = 1/k < 1$ are functions which are mutually inverse.

2.3. Derived quantities in the accumulation and discount laws

In light of the laws defined in (2.1') and (2.2'), we can deduce the following derived quantities[11].

2.3.1. *Accumulation*

As a function of the *initial accumulation factor*

$$\text{iaf} := m(X,Y) \tag{2.10}$$

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them in financial operations, and if the parameters a and m used in such an operation and summarized in z are not reciprocal, a non-zero spread is created.

10 In fact if we assume $z(X,Y)$ to be continuous and partially differentiable everywhere, it follows that: $\frac{\partial z}{\partial X} < 0$, $\frac{\partial z}{\partial Y} > 0\ \forall(X,Y)$. Therefore, the contour curves $z(X,Y)$ = const. are continuous and strictly increasing; they are graphs of functions $Y = \psi(X)$ invertible. In fact, for a theorem on implicit function, it follows that: $\psi'(X) = -\frac{\partial z}{\partial X} / \frac{\partial z}{\partial Y}$, where in the aforementioned hypothesis $\psi(X)$ is continuous and $\psi'(X) > 0$.

11 In this section we will denote with roman capital letters the temporal variables meaning *time* or *epoch* and with small roman letters, variables meaning *length*.

(:= means “equal by definition”) – which measures the multiplicative increment from X to $Y>X$ of the invested capital in X. The factor is “initial” because the date X of investment coincides with the beginning of the time interval (X,Y) on which such an increment is measured. We can also define (see Figure 2.1):

– the *initial interest (per period) rate* (= interest on the unitary invested capital in the time interval from X to $Y>X$) is expressed by

$$\text{iir} := m(X,Y) - 1 \tag{2.11}$$

– the *initial interest (per period) intensity*, expressed by

$$\text{iii} := \{m(X,Y) - 1\}/(Y-X) = \{m(X,X+t) - 1\}/\,t \tag{2.12}$$

where $t = Y\text{-}X > 0$.

Alternatively, still using X as the investment time and imposing $X<Y<Z$, the capital increment is measured on a time interval *(Y,Z) subsequent to X*, then *continuing* with respect to interval (X,Y) without disinvesting in Y, we can then generalize and define continuing factors, rates and intensities in the following way:

– the *continuing accumulation factor* from Y to Z (= accumulated amount in $Z=Y+u$, $u>0$, of the unitary accumulated amount in $Y=X+t$, $t>0$, for the investment started in X) is expressed by

$$\text{caf} := r(X;Y,Z) = m(X,Z)/m(X,Y) = m(X,Y+u)/m(X,Y) \tag{2.13}$$

– the *continuing interest (per period) rate* from Y to Z (= interest for unitary accumulated amount in Y passing from Y to $Z=Y+u$, $u>0$, for the investment started in X) is expressed by

$$\begin{aligned}\text{cir} &:= \text{caf} - 1 = \{m(X,Z) - m(X,Y)\}/m(X,Y) \\ &= \{m(X,X+u) - m(X,Y)\}/m(X,Y)\end{aligned} \tag{2.14}$$

– the *continuing interest (per period) intensity* from Y to $Z = Y+u$, $u>0$ is expressed by

$$\text{cii} := \frac{r(X;Y,Z)-1}{Z-Y} = \frac{m(X,Z)-m(X,Y)}{(Z-Y)\,m(X,Y)} = \frac{m(X,Y+u)-m(X,Y)}{u\;m(X,Y)} \tag{2.15}$$

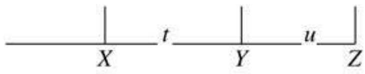

Figure 2.1. *Times in accumulation*

(2.13) is justified if we point out that if the amount K is invested at date X, the accumulated amount in Y has the value $K_Y = K\, m(X,Y)$ while that in Z has the value $K_Z = K \cdot m(X,Z)$. By definition $r(X;Y,Z)$ satisfies $K_Z = K_Y\, r(X;Y,Z)$. For comparison

$$r(X;Y,Z) = K_Z / K_Y = m(X,Z) / m(X,Y)$$

It is obvious that if $X=Y$, (2.13), (2.14) and (2.15) become respectively (2.10), (2.11) and (2.12), i.e. the "continuing" quantities become the "initial" quantity. In symbols: $r(Y;Y,Z) = m(Y,Z)$.

Intensity (2.15) is obtained by dividing the partial incremental ratio of function $m(\xi,\eta)$, considered with $\xi=X$ and respect to η from Y to $Y+u$, by $m(X,Y)$. In the hypothesis that $m(\xi,\eta)$ is partially differentiable with respect to η with a continuous derivative in the interesting interval, the right limit of (2.15) then exists when $u \to 0$, which represents *the instantaneous interest intensity*[12] (implying: *continuing*) in Y of an investment started in X, indicated by $\delta(X,Y)$. Using symbols, where "$\log_e$" is indicated with "ln":

$$\delta(X,Y) = \lim_{u \to 0} \frac{m(X,Y+u)-m(X,Y)}{u\ m(X,Y)} =$$

$$= \left\{ \frac{\partial}{\partial \eta} m(X,\eta) \right\}_{\eta=Y} / m(X,Y) = \left\{ \frac{\partial}{\partial \eta} \ln m(X,\eta) \right\}_{\eta=Y} \qquad (2.16)$$

Working on the variables ξ, η, with $\xi < \eta$, it can be concluded that $\delta(\xi,\eta)$ is the logarithmic derivative (partial with respect to η) of $m(\xi,\eta)$.

12 It can also be called the *interest force* or (but improperly from a dimensional point of view) *instantaneous interest rate.*

Inverting function δ and the derivative operator in (2.16), the important relationship is obtained for continuing accumulated amount (2.13) as a function of the instantaneous intensity[13]:

$$\frac{m(X,Y+u)}{m(X,Y)} = e^{\int_{Y}^{Y+u} \delta(X,\eta)d\eta} \tag{2.16'}$$

2.3.2. *Discounting*

Let X be the final time of a financial operation (for example, the maturity of a credit). Analogously to accumulation, as a function of the *initial discounting factor*

$$\text{idf} := a(X,Y) > 0 \tag{2.17}$$

we can also define (see Figure 2.2):

– the *initial per period discounting rate* (= discount for unitary capital at maturity for the anticipation from X to $Y<X$), given by

$$\text{idr} := 1 - a(X,Y) \tag{2.18}$$

as well as, given $t = X\text{-}Y > 0$:

– the *initial per period discounting intensity*, which can be expressed by:

$$\text{idi} := \{1 - a(X,Y)\}/(X\text{-}Y) = \{1 - a(X,X - t)\}/\, t \tag{2.19}$$

The dynamic expressions for “continuing discount” for an increment of the length of discount are seldom used, but they have meaning in discounting because of the decrease of the present value in relation to the length of anticipation. Therefore, we also define, in relation to the discount, the *continuing per period intensity* as well as the instantaneous intensity, related to time $X>Y>Z$. Indicating by $u>0$ the length of further discount $Z = Y - u$, we define:

13 From (2.16) it follows that, for small u, $m(X,Y)\cdot\delta(X,Y)\Delta u$ linearly approximates $\Delta m = m(X,Y+u) - m(X,Y)$. Furthermore, in the profitable hypothesis of the invested capital, which implies $m(X,X+t)>1$ and increasing with t, the positivity of $\delta(X,\eta)$, $\forall\eta>X$, because of (2.17) and of a well known property of integrals, follows. The opposite is true. A similar conclusion is obtained for the discounting instantaneous intensity, which will be introduced later.

– the *continuing discounting factor from Y to Z* (= present value in $Z<Y$ of the present unitary value in $Y<X$ of the capital at maturity in X, then of amount $1/a(X,Y)$), expressed by:

$$\text{cdf}:= a(X,Z)/a(X,Y) = a(X,Y-u)/a(X,Y) \tag{2.20}$$

– the *continuing discounting rate* from Y to Z (= discount for the anticipation from Y to Z of the present unitary value in Y<X of a capital with maturity in X, then of amount 1/a(X,Y)), expressed by:

$$\text{cdr}:=1-\text{cdf} = \frac{a(X,Y)-a(X,Z)}{a(X,Y)} = \frac{a(X,Y)-a(X,Y-u)}{a(X,Y)} \tag{2.21}$$

– the *continuing discounting intensity from Y to Z*, expressed by:

$$\text{cdi}:=\frac{1-fsp}{Y-Z} = \frac{a(X,Y)-a(X,Z)}{(Y-Z)\ a(X,Y)} = \frac{a(X,Y-u)-a(X,Y)}{-u\ \ a(X,Y)} \tag{2.22}$$

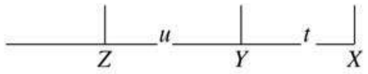

Figure 2.2. *Times in discounting*

Considering the limit as already calculated for the instantaneous interest intensity, it is possible to obtain:

– the *instantaneous discounting intensity* in Y, indicated by $\theta(X,Y)$ and given by:

$$\theta(X,Y)=\left\{\frac{\partial}{\partial\eta}a(X,\eta)\right\}_{\eta=Y} / a(X,Y) = \left\{\frac{\partial}{\partial\eta}\ln a(X,\eta)\right\}_{\eta=Y} \tag{2.23}$$

As $\theta(X,Y)$ is the logarithmic derivative (partial with respect to $Y \le X$) of $a(X,Y)$, by inverting the process we obtain, $\forall Z < Y$,

$$\frac{a(X,Z)}{a(X,Y)} = e^{\int_Y^Z \theta(X,\eta)d\eta} = e^{-\int_Z^Y \theta(X,\eta)d\eta} \tag{2.24}$$

2.4. Decomposable financial laws

2.4.1. *Weak and strong decomposability properties: equivalence relations*

In the case of the financial law of two variables, we consider the meaning and the consequences of the *decomposability* property, which was introduced by Cantelli.

We have decomposability in an accumulation (or discounting) operation when investing (or discounting) a given capital available at time X, we have the same accumulated amount (or present value) in Z, both if we realize and reinvest immediately the obtained value in a intermediate time Y, or if we continue the financial operation. To summarize, decomposability means *invariance of the result with respect to interruptions of the financial operation.*

With reference to the *interest law* $m(X,Y)$, which follows from relation $\approx$, and to the three times X, Y, Z, with $X<Y<Z$, let S_2 be the realized accumulated amount in Y of S_1 invested in X; moreover, let S_3 be the accumulated amount in Z of S_2 immediately reinvested in Y. Instead S'_3 is the accumulated amount Z after only one accumulation of S_1 from X to Z. Due to (2.1')

$$S_2 = S_1\, m(X,Y);\; S_3 = S_2\, m(Y,Z);\; S'_3 = S_1\, m(X,Z)\,. \tag{2.25}$$

If

$$S_3 = S'_3,\ \forall(S_1,\ X<Y<Z) \tag{2.26}$$

the interest law is decomposable. It follows from (2.25) that (2.26) is equivalent to

$$m(X,Y)\, m(Y,Z) = m(X,Z) \tag{2.27}$$

which expresses the decomposability condition for an interest law in terms of accumulation factors.

In the same way, referring to the *discount law* $a(X,Y)$ following $\approx$ and recalling (2.2'), if $X > Y > Z$ we can define the following discounted values starting from S_1, payable in X:

$$S_2 = S_1\, a(X,Y);\; S_3 = S_2\, a(Y,Z);\; S'_3 = S_1\, a\,(X,Z) \tag{2.28}$$

If

$$S_3 = S'_3, \forall(S_1, X>Y>Z) \tag{2.29}$$

the discount law is decomposable and because of (2.28) the decomposability condition for this law can be written as

$$a(X,Y)\, a(Y,Z) = a(X,Z) \tag{2.30}$$

Until now, we have defined in *weak* form the decomposability of single laws in accumulation or discounting operations, considering the times *X, Y, Z* in increasing or decreasing order. This signifies that we require the *prospective transitivity* or respectively the *retrospective transitivity* to the indifference relations, which give rise to the laws.[14] In this case we will talk of *weak decomposability.*

If instead the previous considerations are related to an exchange law following an indifference relation $\approx$ and expressed by the factors $z(X,Y)$ defined in (2.5'), we can think of extending the decomposability relation in (2.25) and (2.26) for any order of payment times. So the relation $\approx$ satisfies the *strong decomposability* property, which bi-implies

$$\{(X,S_1) \approx (Y,S_2)\} \cap \{(Y,S_2) \approx (Z,S_3)\} \Rightarrow (X,S_1) \approx (Z,S_3), \forall(X,Y,Z) \tag{2.31}$$

and then the following condition on the exchange factors:

$$z(X,Y)\, z(Y,Z) = z(X,Z),\ \forall(X,Y,Z) \tag{2.32}$$

The following result holds:

THEOREM A.– *If and only if for the exchange law the strong decomposability is valid, the relation $\approx$ is reflexive, symmetric and transitive, then it is an equivalence relation, which we denote by $\mathcal{E}$.*

Proof

Sufficiency: the strong decomposability implies (2.32); putting $Y = Z$ we obtain the reflexivity; putting $Z = X$ we obtain the symmetry; the transitivity is obvious.

Necessity: if $\approx = \mathcal{E}$, the unitary amount in *X* is exchangeable with $z(X,Z)$ in *Z* and also with $z(X,Y)$ in *Y*, which is exchangeable with $z(X,Y)\, z(Y,Z)$ in *Z* (whatever order

14 Let us recall that a binary relation $\mathcal{R}$ on a set $\mathcal{H}$ satisfies the *transitivity property* if $(a\,\mathcal{R}\,b) \cap (b\,\mathcal{R}\,c) \Rightarrow a\,\mathcal{R}\,c,\ \forall\ a,b,c \in \mathcal{H}$.

may be among X, Y and Z because of the symmetry property); then (2.32) is also valid if $X = Z$ or if $Y = Z$.

Note: this argument could be developed, in a more formally complicated, but equipollent way, based on relation (2.31).

Considering the relation between weak decomposability (WD) and strong decomposability (SD), it is obvious that the condition of SD implies WD, when X, Y, Z are in increasing or decreasing order from which there are only accumulation or discounting respectively. However, the WD does not imply SD in other cases, when both an accumulation and a discounting occur together. Then, if SD holds, the properties of an equivalence are immediately verified. In fact, considering $X < Z < Y$ (analogously we could consider $X > Z > Y$), the SD expressed by (2.32) gives rise to

$$m(X,Y)\,a(Y,Z) = m(X,Z) \tag{2.33}$$

and the WD following the SD also implies $m(X,Z)\,m(Z,Y) = m(X,Y)$, or, for (2.33), $m(Z,Y) = m(X,Y)/m(X,Z) = 1/a(Y,Z)$, or also that

$$m(Z,Y)\,a(Y,Z) = 1,\ \forall Y<Z \tag{2.34}$$

Then, because of the generic choice of times, m and a are conjugate laws, the financial relation is symmetric, as well as transitive, but also reflexive (it is enough to impose $Y = Z$ in (2.33) obtaining $a(Z,Z) = 1$ and then for (2.34), $m(Z,Z) = 1$). Therefore, the relation is an equivalence; the opposite also holds.

Let us summarize as follows. Given an indifference relation $\approx$ in the hypothesis of proportional amount, the strong decomposability, expressed by (2.32) for the exchange factor $z(X,Y)$, implies that $\approx$ is reflexive, symmetric and transitive, and then it is an equivalence indicated by $\mathcal{E}$. In this case, the derived interest and discount laws are decomposable and conjugated to each other.

EXAMPLE 2.3.– An investor with liquid assets invests the amount S_1 at time X until time Y in a term deposit. A prospectively decomposable accumulation law with accumulation factor $m(X,Y)$ is applied and a refund of $S_2 = S_1\,m(X,Y)$ is expected. At time Z (with $X < Z < Y$) the investor needs liquidity, but he cannot use the capital in the term deposit; therefore, the accumulated amount, given by

$$S'_3 = S_1\,m(X,Z) = S_2\,\frac{m(X,Z)}{m(X,Y)} = \frac{S_2}{m(Z,Y)}$$

is not available (as when the capital is invested in a bank account); it is only possible to transfer the credit S_2 with a bank advance, applying a retrospectively decomposable discounting law to *a(Y,Z)*. In practice, in these cases the laws *m(Z,Y)* and *a(Y,Z)* are not conjugated, i.e. (2.34) does not hold. Thus, we do not have strong decomposability of the resulting exchange law, even if the laws *m* and *a* are weakly decomposable. We usually have $a(Y,Z) < 1/m(Z,Y)$, i.e. the cost for discount is greater than that resulting from applying the conjugate law of that regulating the deposit. It follows that $S_3 < S'_3$ and $S'_3 - S_3$ is the cost due to the locking up of capital S_1 until Y. The SD would cause $S'_3 = S_3$ and would avoid such cost.

2.4.2. *Equivalence classes: characteristic properties of decomposable laws*

Based on theorem A, if an indifference relation $\approx$ gives rise to a strongly decomposable exchange law, it is an equivalence relation $\mathcal{E}$[15] between all elements *(T,S)* of the set $\mathcal{H}$ of supplies, which makes it possible to separate such supplies into equivalence classes. Each class is made up of financially equivalent supplies, but which are indifferent. However, two supplies in different classes are not equivalent because it is possible to express a judgment of strong preference. Each class is characterized by an *abstract*, made up of the intrinsic financial value of its supplies.

By geometrically representing the supplies *(T,S)* on the plane *OTS*, a class of equivalent supplies is identified by a curve, a locus of points $P \equiv [T,S]$, corresponding to equivalent supplies. The infinite curves do not have common points. In addition:

1) for each point in the plane there is one and only one curve, which is a locus of equivalent points;

2) such curves are the graph of functions $S = \varphi(T)$ (continuous and differentiable, under suitable hypotheses) and, if the postulate on money return holds, increasing where positive, decreasing where negative.

The classes of equivalent supplies on the basis of an SD, i.e. the elements of the quotient set $\mathcal{H}/\mathcal{E}$, form a *totally ordered set*, because the elements of each couple are comparable for a weak preference judgment $\succeq$, using the meaning specified in section 1.2. Moving monotonically towards the classes (= curve in the plane *OTS*), the intrinsic financial value of the supplies improves in one sense (but gets worse in

15 It is well known that equivalence relations E on the elements of a set H make it possible to stratify these elements in equivalence classes, such that each element is only in one class. Each class is characterized by an *abstract* common to its elements, indicating by *quotient set* H/E the set whose elements are the *abstracts*.

the other sense)[16]. It follows that the SD laws, on the basis of stratification in equivalence classes, allow a global, rather than just local, comparison between

16 We set out the definition of some properties that are applied in the set of financial supplies. Let $\mathcal{H}$ be a set and $\mathcal{R}$ a binary relation between elements a, b, c, ...$\in \mathcal{H}$. The following properties can hold for $\mathcal{R}$ (where = means coincidence between elements, ~ means negation, $\cup$ means union or logic sum and $\cap$ means intersection or logic product):
1) *reflexive property*: $a\mathcal{R}\,a$, $\forall a \in \mathcal{H}$; 2) *symmetric property*: $a\mathcal{R}\,b \rightarrow b\mathcal{R}\,a$, $\forall\, a,b \in \mathcal{H}$; 3) *transitive property*: $(a\mathcal{R}\,b)\cap((b\mathcal{R}\,c) \rightarrow a\mathcal{R}\,c$; $\forall a,b,c \in \mathcal{H}$; 4) *non-reflexive property*: $\sim(a\mathcal{R}\,a)$, $\forall a \in \mathcal{H}$; 5) *anti-symmetric property*: $(a\mathcal{R}\,b)\cap(b\mathcal{R}a) \rightarrow a=b$; $\forall a,b \in \mathcal{H}$; 6) *asymmetric property*: $a\mathcal{R}\,b \Rightarrow \sim(b\mathcal{R}\,a)$, $\forall a,b \in \mathcal{H}$; 7) *completeness property*: $(a\mathcal{R}\,b)\cup(b\mathcal{R}\,a)$ is certainly verified, i.e. at least one of $(a\mathcal{R}\,b)$ and $(b\mathcal{R}\,a)$, $\forall a\neq b \in \mathcal{H}$ holds.

We have already talked about the first three properties, pointing out that a binary relation between elements of $\mathcal{H}$ is an equivalence relation $\mathcal{E}$ if for every choice of elements the symmetric, reflexive and transitive properties hold. When $\forall a,b \in \mathcal{H}$ it is verified that $(a\mathcal{R}\,b) \cup\sim(a\mathcal{R}\,b)$, and then $(a\mathcal{R}\,b)$ is an event, in the logic meaning, referred to elements of $\mathcal{H}$. We give the following definition regarding ordering. A binary relation $\mathcal{R}$ on the set $\mathcal{H}$ is called a *relation of partial order* if for each element in $\mathcal{H}$ the reflexive, anti-symmetric and transitive properties hold. $\mathcal{H}$ is then said to be *partially ordered.* With this hypothesis, if all elements of $\mathcal{H}$ are comparable two by two (= completeness property), then the relation is called *of total order* and $\mathcal{H}$ is said to be *totally ordered.* A binary relation $\mathcal{R}$ on the set $\mathcal{H}$ is said to be *almost ordered or preordered (total or partial*, if it is comparable or not) if the reflexive or transitive properties hold when it is then called *almost ordered (partially or totally).* Briefly, an *order relation* $\mathcal{O}$ brings to a classification which do not consider “equal elements” while a *almost order relation* $\mathcal{QO}$ allows “equal elements”.

Note that if on the set $\mathcal{H}$ a total $\mathcal{QO}$ relation holds, the completeness relation is satisfied, i.e. however chosen $b\in \mathcal{H}$, $\forall a \in \mathcal{H}$, $a\neq b$, it certainly satisfied that $(a\mathcal{QO}b)\cup(b\mathcal{QO}a)$. Given that $(a\mathcal{QO}\,b)\cup(b\mathcal{QO}\,a) = [(a\mathcal{QO}\,b)\cap((b\mathcal{QO}\,a)]\cup[(a\mathcal{QO}\,b)(\sim(b\mathcal{QO}\,a)]\cup[\sim(a\mathcal{QO}\,b)((b\mathcal{QO}\,a)]$ and that the three possibilities written between square brackets in the second term are incompatible, they make a partition. More briefly, the completeness derived from the totality of $\mathcal{QO}$ is equivalent to the possibility of the realization of $\sim(a\mathcal{QO}\,b)\cap\sim(b\mathcal{QO}\,a)$.

Let us now consider the equivalence relation $\mathcal{E}$, such that $a\mathcal{E}\,b$ if the first possibility is true i.e. $(a\mathcal{QO}\,b)\cap(b\mathcal{QO}\,a)$; in such a case we write $a\approx b$. If the second or third possibility is true, we write respectively $a\prec b$ and $b\prec a$. Relation $\prec$ (or $\succ$) is said to be a *(strong) preference*, characterized by the asymmetric property. Writing $a\succ b$ is equivalent to $b\prec a$. In conclusion, as a consequence of the relation $\mathcal{QO}$ in $\mathcal{H}$, of the three possibilities, $a\approx b$, $a\prec b$, $a\succ b$, one and only one is verified. With a fixed $\mathcal{E}$, the quotient set $\mathcal{H}/\mathcal{E}$, i.e. the set of the supplies, i.e. due to transitivity they make it possible to extend to any number of supplies on the plane *OTS* the preference or indifference relations introduced in Chapter 1 with respect to a given supply.

It is easy to give a method for such a comparison, verifying the existence of total order in $\mathcal{H}$. It is enough to identify the classes using supplies that have the same maturity T_0; then class α identified by (T_0,S'_0) is preferred to class β identified by (T_0,S''_0) if $S'_0 > S''_0$; β is preferred to α if $S'_0 < S''_0$; α and β are equivalent if $S'_0 = S''_0$.

Let us consider some characteristic properties of decomposable laws of two variables, which proceed from the following theorems.

THEOREM B.– *Referring to definitions* (2.10) *and* (2.13)*, an interest law is weakly decomposable if and only if, for each choice of subsequent times X<Y<Z, the initial accumulation factor from Y to Z is equal to the continuing accumulation factor from Y to Z of an accumulation started in X. In symbols: r(X;Y,Z)=r(Y;Y,Z)=m(Y,Z). Therefore, the decomposability implies independence of r(X;Y,Z) from the time of investment, and vice versa. There is an analogous condition in relation to the discount factors* (2.17) *and* (2.20)*, for each choice of time X>Y>Z holds for a weakly decomposable discount law.*

THEOREM C.– *An interest law is weakly decomposable if and only if the instantaneous intensity* $\delta(X,T)$*, continuous by hypothesis, does not depend on the initial time X but only on the current time T. The analogous condition on the intensity* $\theta(X,T)$ *holds for a weakly decomposable discount law. Under the same condition necessary and sufficient on the instantaneous intensity of interest and*

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equivalence classes with respect to $\mathcal{E}$ of the elements in $\mathcal{H}$, results totally ordered because between the classes {a}, {b} identified by a, b only one relation holds: {a}={b}, $\{a\} \prec \{b\}$, $\{a\} \succ \{b\}$. To summarize, an almost order relation (total) on $\mathcal{H}$ induces an equivalence relation $\mathcal{E}$ and then an order (total) relation on $\mathcal{H}/\mathcal{E}$.

In financial applications it follows that if the exchange law applicable to the supplies $(T,S) \in \mathcal{H}$ is strongly separable and then follows from an equivalence relation $\mathcal{E}$, then:

1) There is an *almost order* between each supply $\in \mathcal{H}$ (total if the law is applicable to all supplies) where between two supplies or there is indifference or one is preferred (strongly). There is then the possibility of "equals" or indifference.
2) There is *order* (total in the same hypothesis) between supply equivalence classes, elements of $\mathcal{H}/\mathcal{E}$, where between two different classes there is always a strong preference relation, regarding each pair of supplies each taken in a class. In formula, $\{a\} \prec \{b\} \Rightarrow a \prec b$, $\forall$ ($a \in \{a\}$, $b \in \{b\}$).

discount, a strong decomposability of an exchange law specified by the factor identified by (2.5') *which satisfies* (2.9) *can be verified.*

THEOREM D.– *An exchange law specified by the factor identified by* (2.5') *which satisfies* (2.9) *is strongly decomposable if and only if there exists an increasing function h(T) such that*

$$z(X,Y) = \frac{h(Y)}{h(X)} \quad , \ \forall (X,Y) \tag{2.35}$$

Given z(·) = m(·), (2.35) *∀(X≤Y) gives a WD condition for an interest law (= of prospective transitivity for ≈); furthermore,* (2.35)*, ∀(X≥Y), and given z(·) = a(·), is WD condition for a discount law (= of retrospective transitivity for ≈). If ≈ is not symmetric, i.e.* (2.9) *is not valid, we have weak decomposability of interest and the discount law is not conjugated following ≈ if and only if there exist two different functions* $h_1(T)$ *and* $h_2(T)$ *such that* (2.35) *holds where:* $h(T) = h_1(T)$ *if* $X \le Y$*;* $h(T) = h_2(T)$ *if* $X > Y$ [17].

Briefly, theorems C and D show that: 1) a characteristic property of strongly decomposable exchange laws is the coincidence of interest and discount intensity in

17 The proofs of theorems B, C and D are as follows:

– theorem B is proved by noticing that, with respect to the interest (or discount) laws, the equality between *m(Y,Z)* and *m(X,Z)/m(X,Y)* (or between *a(Y,Z)* and *a(X,Z)/a(X,Y)*) bi-implies (2.28) or (2.31);

– theorem C is proved, with respect to interest laws, by noticing that because of (2.17) and of theorem B the decomposability of law *m* is equivalent to the identity chain:

$$e^{\int_Y^Z \delta(X,\eta)d\eta} = \frac{m(X,Z)}{m(X,Y)} = m(Y,Z) = e^{\int_Y^Z \delta(Y,\eta)d\eta}, \forall (X<Y<Z)$$

which, because of the arbitrariness of time, bi-implies $\delta(X,\eta) = \delta(Y,\eta)$, i.e. because of the same arbitrariness, an intensity depends only on current time. An analogous proof holds in regard to the condition on the intensity $\theta(X,T)$ to have decomposability of the discount law, $\forall (X>Y>Z)$, and on the intensity condition $\delta(X,T) = \theta(X,T) = \delta(T)$ to have strong decomposability of the exchange law $z(X,T)$, $\forall (X,Y,Z)$;

– theorem D for exchange law is proved by noticing that:

sufficient condition: if there is *h(T)* verifying (2.30), clearly $z(X,X) = 1$, $\forall X$, and then (2.9) and (2.33) hold so that $\approx = \mathcal{E}$ and the exchange law is strongly decomposable,

necessary condition: if *z (X,Y)* identifies a strongly decomposable law, because of theorem C the interest and discount intensity are expressed by the same function $\delta(T)$ and the requested function $h(T)$, which is clearly defined regardless of a multiplicative constant, has the dimension and meaning of an amount valued in *T* and must satisfy the differential equation: $h'(T) = \delta(T)h(T)$, where the general expression $h(T) = k\, e^{\int_{T_0}^T \delta(\eta)d\eta}$ is assumed as having the meaning of valuation in *T*, based on the exchange law *z* of the amount *k* dated at time T_0. Theorem D regarding conditions of weak decomposability is an immediate corollary.

a function δ(*T*) which depends only on current time; 2) the exchange factor of a strongly decomposable law assumes the characteristic form

$$z(X,Y)=e^{\int_X^Y \delta(X,\eta)d\eta} \tag{2.36}$$

EXAMPLE 2.4.– Give the following accumulation law

$$m(X,Y)=e^{0.05(Y-X)+0.002(Y+X)(Y-X)}$$

using an instantaneous intensity $\delta(t) = 0.05 + 0.004\,t$, a function only of the current time t, where $m(X,Y)$ is a decomposable law.

Let us verify the decomposability using (2.27). We obtain

$$m(X,Z)=e^{0.05(Z-X)+0.002(Z^2-X^2)}\,;\; m(X,Y)=e^{0.05(Y-X)+0.002(Y^2-X^2)}$$

$$m(Y,Z)=e^{0.05(Z-Y)+0.002(Z^2-Y^2)}$$

then (2.6) $\forall(X<Y<Z)$ holds.

If we put: $X = 1$; $Y = 5+\frac{5}{12} = 5.417$; $Z = 6+\frac{1}{12} = 6.083$, it results in

$$m(X,Y) = e^{0.05\cdot 4.417+0.002\cdot 28.344} = e^{0.277538} = 1.319876$$

$$m(Y,Z) = e^{0.05\cdot 0.666+0.002\cdot 7.659} = e^{0.048618} = 1.049819$$

$$m(X,Z) = e^{0.05\cdot 5.083+0.002\cdot 36.003} = e^{0.326156} = 1.385632$$

and then (summing the exponents of *e*) (2.6) is verified. Even the alternative expression following theorem B is verified as

$$r(X;Y,Z)=\frac{m(X,Z)}{m(X,Y)}=\frac{1.385632}{1.319876}=1.049819 = m(Y,Z)=r(Y;Y,Z)$$

EXAMPLE 2.5.– Given, with $Y<Z$,

$$m(Y,Z) = 1+1.06^Z - 1.06^Y$$

satisfying $m(Y,Y)$=1, increasing with Z, decreasing with Y, resulting in: $m(0,Z)$ = 1.06^Z. Put S_1=1,450, $Y = 5+\frac{5}{12} = 5.417$, $Z = 6+\frac{1}{12} = 6.083$, it follows that

$$m(Y,Z) = 1 + 1.425396 - 1.371140 = 1.054256$$

and then: S_2 = 1,528.67; initial per period rate = 0.054256; initial per period intensity = 0.081465 years^{-1}.

Given X = 1 it follows, in continuing terms, that:

$$r(X;Y,Z) = \frac{1+1.425396 - 1.06}{1+1.371140 - 1.06} = \frac{1.365396}{1.311140} = 1.04138 \neq m(Y,Z).$$

This financial law is not decomposable. In addition:

– the continuing per period rate is 0.041381;

– the continuing per period intensity is 0.062078 years^{-1}.

2.5. Uniform financial laws: mean evaluations

2.5.1. *Theory of uniform exchange laws*

The hypothesis of *uniformity* (or *homogenity*) *in time* is common in financial practice. In formal terms, an indifference financial relation ≈ is *uniform in time* if:

$$(X,S_1) \approx (Y,S_2) \Rightarrow (X+h,S_1) \approx (Y+h,S_2),\ \forall h \tag{2.37}$$

that is, an indifference relation is not changed by a time translation (i.e. moving X and Y of the same time interval forwards or backwards), as long as the payment times remain in the applicability interval of the financial law.

Assuming the proportionality of amounts, because of (2.37) for the exchange factor $z(X,Y) = S_2/S_1$ the following property is worth:

$$z(X,Y) = z(X+h,Y+h),\ \forall h \tag{2.38}$$

To summarize: *a uniform relation is characterized by the property that the exchange factor does not change with a rigid time translation such that the time difference $Y - X = (Y+h) - (X+h)$ does not change.*

It follows for the corresponding financial law (which we will call *uniform*) that

$$z(X,Y) \equiv g\,(Y\text{-}X) \equiv g\,(\tau) > 0,\ \forall\tau \tag{2.39}$$

that is, *in a uniform law the exchange factor depends only on the duration (with sign)* $\tau=Y\text{-}X$ *of the financial operation and not just on the times X,Y of the beginning and the end of the operation, considered separately*.

If the relation ≈ is *uniform* and also *symmetric*, the couples of conjugated interest and discount laws are expressed by the factors $g(\tau)$ and $g(-\tau)$[18] satisfying

$$g(\tau)\, g(-\tau) = 1,\ \forall\tau \tag{2.40}$$

If the exchange law $z(X,Y)$ is uniform on time, the contour curves $z(X,Y) = const.$ are lines parallel to the bisector $Y=X$. Furthermore, if ≈ is also symmetric, considering geometrically (2.40), the increasing graph of $g(\tau)$ is such that the opposite values of τ correspond with the reciprocal values of $g(\tau)$. Such factors remain constant respectively on parallel lines equidistant of the bisector $\tau = 0$, from which $z(X,X) = g(0) = 1$ follows.

Often the accumulation and discount factor, instead of being considered unified through $g(\tau)$, are considered separately and expressed as a function of the (absolute) duration $t = |Y–X| = |\tau|$.

Obviously we have:

$$t = \tau\ ,\ \text{if}\ \tau > 0;\ t = -\tau\ ,\ \text{if}\ \tau < 0.$$

We can then put a correspondence between a *uniform* relation ≈, which is characterized by a exchange factor $g(\tau)$, defined $\forall\tau$, and two laws, the former of interest, expressed by an *accumulation factor u(t)*, the latter of discount, expressed by a *discount factor v(t)*, both defined $\forall t \geq 0$ in the following way:

$$\begin{cases} u(t) = g(t) = g(\tau)\ , \quad if\ \tau = t > 0 \\ \qquad u(0) = v(0) = g(0) = 1 \\ v(t) = g(\tau) = g(-t)\ ,\ if\ \tau = -t < 0 \end{cases} \tag{2.41}$$

In (2.41), the second equation express the reflexive property of ≈; the first and the third equation express respectively the exchange factor in accumulation and discount. By assuming the usual hypothesis of onerous nature of a loan, $u(t) \geq 1$ is a

18 More precisely, in an accumulation law, the result is $\tau = Y\text{-}X > 0$ and $g(\tau)$ is *the accumulation factor*, whereas $g(-\tau) = 1/g(\tau)$ is the conjugate *discount factor* from *Y* to *X*. However, in a discount operation, the result is $\tau = Y\text{-}X < 0$ and $g(\tau)$ is *the discount factor* while $g(-\tau) = 1/g(\tau)$ is the conjugated *accumulation factor* from *Y* to *X*.

strictly increasing function of the duration t, and $v(t)$, subject to $0 < v(t) \leq 1$, is a strictly decreasing function of t.

If $\approx$ is also *symmetric*, from (2.40) and (2.41) it follows that:

$$u(t)\, v(t) = 1, \ \forall t > 0 \tag{2.42}$$

that is, the accumulation and the discount factors for a fixed duration t are reciprocal.

It is useful at this point to adopt for the uniform laws and for the exchange factors $u(t)$ and $v(t)$ the definitions and positions introduced for the factors $m(X,Y)$ and $a(X,Y)$. The following table is then obtained[19].

FACTORS, RATES AND INTENSITIES FOR UNIFORM LAWS			
Financial quantity		*Interest laws*	*Discount laws*
I)	*initial accumulation factor* for duration t	$u(t)$	$v(t)$
II)	*initial rate* for duration t	$u(t) - 1$	$1 - v(t)$
III)	*initial intensity* for duration t	$\frac{u(t) - 1}{t}$	$\frac{1 - v(t)}{t}$
IV)	*continuing accumulation factor* for the subsequent duration h after t	$\frac{u(t+h)}{u(t)}$	$\frac{v(t+h)}{v(t)}$
V)	*continuing rate* for the subsequent duration h after t	$\frac{u(t+h)}{u(t)} - 1$	$1 - \frac{v(t+h)}{v(t)}$
VI)	*continuing intensity* for the subsequent duration h after t	$\frac{u(t+h) - u(t)}{h\ u(t)}$	$\frac{v(t) - v(t+h)}{h\ v(t)}$
VII)	*instantaneous intensity* in t (*)	$\delta(t) = \frac{u'(t)}{u(t)}$	$\theta(t) = -\frac{v'(t)}{v(t)}$
(*) (VII) is the limit case of (VI) when $h \to 0$ and assumes the derivability of exchange factors $u(t)$ and $v(t)$. Prime means differentiation. For simplicity, intensities are indicated with the same symbols δ and θ used for those connected with law of two variables.			

Table 2.1. *Factors, rates and intensities for uniform laws*

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19 We notice that because of the invariance with translation following (2.39), it is possible and convenient to choose the time origin as X, the "beginning" time of the operation, and to measure time forwards (in interest laws) or backwards (in discount laws) for a time interval of length t.

From definition VII in Table 2.1, which expresses $\delta(t)$ and $-\theta(t)$ as logarithmic derivatives of $u(t)$ and $v(t)$, by inversion it follows that:

$$u(t) = e^{\int_0^t \delta(z)\,dz}\ ;\ v(t) = e^{-\int_0^t \theta(z)\,dz} \tag{2.43}$$

If the uniform interest and discount laws are conjugated (i.e. in the symmetry hypothesis), it results in $\delta(t) = \theta(t)$. In fact, it validates the theorem.

THEOREM.– *The necessary and sufficient condition in order for* (2.42) *to hold is the equality* $\delta(t) = \theta(t)$, $\forall t \geq 0$.

Proof:

Necessity: if (2.42) holds, it follows that $\forall t \geq 0$: $\ln u(t) = -\ln v(t)$ and, differentiating, $\delta(t) = \theta(t)$.

Sufficiency: if $\delta(z) = \theta(z)$, $\forall z \geq 0$, for (2.43) it follows, $\forall t \geq 0$, that

$$u(t)\,v(t) = e^{\int_0^t [(\delta(z)-\theta(z)]\,dz} = 1$$

because the integrand function is identically zero in the interval $(0,t)$.

Examples and applications of laws uniform in time will be shown in Chapter 3.

2.5.2. *An outline of associative averages*

Let us recall the concept of mean, as introduced by Chisini and developed by de Finetti[20], from which the mean of quantities $x_1, x_2, ..., x_n$ with respect to a quantity $y = f(x_1, x_2, ..., x_n)$, which depends univocally on $x_1, x_2,...,x_n$ by the function f, is a value $\hat{x}$ such that:

$$f(\hat{x},\hat{x},...,\hat{x}) = f(x_1, x_2, ..., x_n) \tag{2.44}$$

where if $x_1, x_2,...,x_n$ are replaced by $\hat{x}$, f remains unchanged. In such a way the individuation of mean, which has a summarizing meaning, depends on the considered problem which constitutes a choice criterion.

20 See, for example, de Finetti (1931); Volpe di Prignano (1985).

A mean is said to be *associative* when the same result is obtained, averaging out the given quantities (each with its *weight*) or averaging out the partial averages of their subgroup (each with the total weight of the subgroup). The consequent "associative property" is verified by the center of mass of a distribution of masses concentrated on the point of a line, a center whose abscissa $\bar{x} = \sum_h p_h x_h / \sum_h p_h$ is the weighted arithmetic mean[21] of the abscissas x_h where the masses are put, with weights p_h corresponding to the masses. It can be proved (see the *Nagumohy Kolmogoroff-de Finetti theorem*) that, given the distribution (x_h, p_h), $(h = 1, ..., n)$, the set of its associative averages coincides with the set of transformations of the arithmetic mean through a function $q(x)$ chosen in the class of continuous and strictly monotonic functions. In other words, with $q(x)$ continuous and strictly decreasing or increasing, the number $\hat{x}_q$, solution of the following equation in x

$$q(x) = \sum_h p_h \, q(x_h) / \sum_h p_h \tag{2.45}$$

is an associative average of the values x_h with weights p_h and all the others can be obtained by varying $q(x)$ in the class specified above. Since $q(x)$ has an inverse function $q^{-1}(x)$, we univocally obtain

$$\hat{x}_q = q^{-1}\left(\sum_h p_h \, q(x_h) / \sum_h p_h\right) \tag{2.46}$$

$\hat{x}_q$, called q-average, is invariant for linear transformation on $q(x)$, because it follows from (2.45) that

$$a \, q(x)+b = \sum_h p_h [a \, q(x_h) + b] / \sum_h p_h$$

The more important averages used in applications are associative.[22]

The following properties hold:

1) the geometric mean can be obtained as the limit of the power mean when $k \to 0$;

21 If the weights are all equal, the mean is called "simple".
22 Let us recall the mean of powers of order k, with transformation function $q(x) = x^k$ (the arithmetic mean for $k = 1$, the quadratic mean for $k = 2$, and the harmonic mean for $k = -1$), the geometric mean for $q(x) = \log x$, and the exponential mean for $q(x) = e^{cx}$.

2) with the same data, power means with exponent k give values increasing with k;

3) the inequality between $\hat{x}_q$ and $\bar{x}$ depends on the feature of $q(x)$, resulting:

– $\hat{x}_q > \bar{x}$, if $q(x)$ is increasing convex or decreasing concave,

– $\hat{x}_q < \bar{x}$, if $q(x)$ is increasing concave or decreasing convex.

The concavity and convexity are, as usual, downwards.

The aforementioned properties are shown in Figure 2.3, which explains the calculation of a simple associative average of two elements.

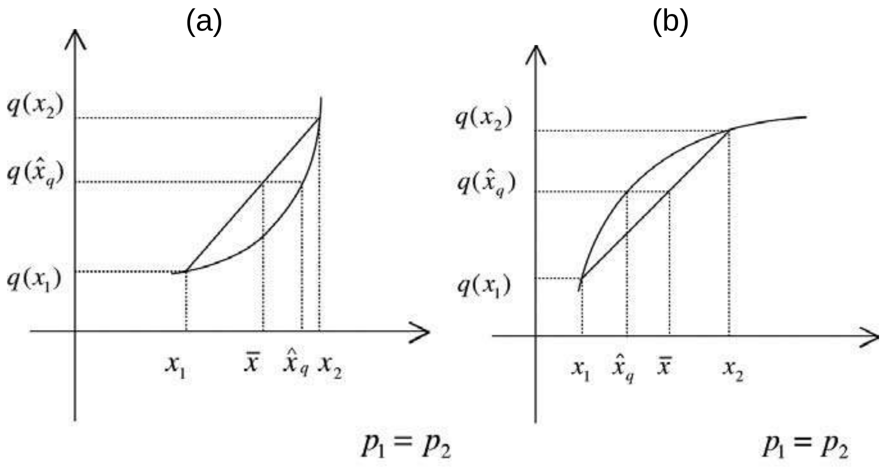

Figure 2.3.a *Associative average with convex q(x)*

Figure 2.3.b *Associative average with concave q(x)*

2.5.3. *Average duration and average maturity*

Let us consider a financial relation $\approx$ expressed by a law with an always positive intensity, and suppose that the exchange factors $q(t)$ consequent to $\approx$ are continuous and strictly monotonic of the duration.

Let us also consider the following problem: given the amounts K_1, K_2, ..., K_n accumulated or discounted according to the same exchange law $q(t)$ and the respective durations t_1, t_2, ..., t_n, we want to find the duration $\hat{t}_q$ of investment (or discount) according to $q(t)$ of the amount $K = \sum_h K_h$ so as to have the same interest (or discount) obtainable as with the original operation on the n amounts K_1, $K_2,..., K_n$[23].

Under these assumptions the value $\hat{t}_q$ is univocally determined and is called the *average length* (or *average maturity,* choosing 0 as starting point) of the operation. This makes it possible, having fixed the starting time T_0 (i.e. the beginning of the investment or maturity of the amount to be discounted), to find the *average maturity* $T_1 = T_0 + \hat{t}_q$ (in accumulation) or $T_1 = T_0 - \hat{t}_q$ (in discounting).

The average length $\hat{t}_q$ depends on the choice of $q(t)$ and can be found by using (2.46) with $\hat{x}_q = \hat{t}_q$, $p_h = K_h$, $x_h = t_h$. Based on the financial meaning, in accumulation the interest obtained with the n investments based on the factor $q(t) = u(t)$ for the given times t_h is $\sum_{h=1}^{n} K_h [u(t_h) - 1]$, while that obtained with only one investment of $\sum_{h=1}^{n} K_h$ for the time t is $(\sum_{h=1}^{n} K_h)[u(t) - 1]$; these values are the same if $t = \hat{t}_q$. The position is analogous in discounting with $q(t) = v(t)$. This proves the following theorem.

THEOREM.– *In a financial operation of investment or discount of more than one amount with different durations* t_h, *the average length* $\hat{t}_q$ *is associative and coincides with the q-average of the lengths, weighted with the amounts* K_h, *where the transformation function q(t) coincides with the factor u(t) or v(t), respectively in an accumulation or discount operation.*

2.5.4. *Average index of return: average rate*

Let us consider the following problem of averaging. Let us invest for the same duration t the amounts C_1, $C_2,...,C_n$ by using accumulation laws (for simplicity following the same regime) with different returns, based on the factors $u_1(t),..., u_n(t)$. We want to find the accumulation factor that leaves the total interest unchanged for the same duration t. The solution is:

$$\hat{u}(t) = \sum_{h=1}^{n} C_h u_h(t) / \sum_{h=1}^{n} C_h \tag{2.47}$$

23 The financial operations with n>2 amounts, which are called *complex*, will be discussed in Chapter 4.

The same result is obtained for a discount of length t, with factors $v_h(t)$ applied at maturity to the amounts M_h.

The following theorem is then proved.

THEOREM.– *Applying different exchange factors to different amounts for the same duration t, in accumulation or discount operations, the factor which does not change the returns is the arithmetic average of factors weighted with the amounts.*

If the accumulation factors $u_h(t)$ can be expressed $\forall t$ with the same invertible function $q(i_h;t)$ of the interest rates i_h ($h = 1,...,n$), the *mean rate* $\hat{i}_q(t)$ is defined by

$$\hat{i}_q(t) = q^{-1}\left\{\sum_{h=1}^{n} C_h\, q(i_h;t) / \sum_{h=1}^{n} C_h\right\} \tag{2.48}$$

In the same way, the mean rate $\hat{d}_q(t)$ of the discount rates d_h ($h = 1,...,n$) is defined for discounting, using in (2.48) the discount factors $q(d_h;t)$ instead of the accumulation factors $q(i_h;t)$ and the capitals at maturity M_h instead of the invested capitals C_h.

2.6. Uniform decomposable financial laws: exponential regime

We have already shown the practical importance of uniform financial laws. In relation to a *financial regime* – defined as a set of financial laws, based on a common feature and identified in the set by a parameter – it is important to investigate the existence and the properties of regimes which are decomposable and, at the same time, uniform. Hence, given that the financial laws

$$u(t) = e^{\delta t} \quad ; \quad v(t) = e^{-\theta t} \tag{2.49}$$

are called *exponential laws* and, by varying the parameters δ and θ, they constitute the *exponential regime* (often considered in symmetric hypothesis i.e. $\delta = \theta$), the following theorem holds.

THEOREM.– *The exponential regime, characterized by intensities constant in time, is the only one to be decomposable and uniform.*

Proof:

1) If $X \leq Y \leq Z$, given $Y - X = t_1$, $Z - Y = t_2$ and then $Z - X = t_1+t_2$, if $\approx$ is uniform, it follows that: $m(X,Y) = u(t_1)$; $m(Y,Z) = u(t_2)$; $m(X,Z) = u(t_1+t_2)$. From (2.27), because of decomposability, we obtain the following *characterization of decomposable and uniform interest laws:*

$$u(t_1)\,u(t_2) = u(t_1+t_2);\ \forall(t_1 \geq 0,\ t_2 \geq 0) \tag{2.50}$$

2) If $X \geq Y \geq Z$, given $X - Y = t_1$, $Y - Z = t_2$ and then $X - Z = t_1+t_2$, if $\approx$ is uniform, it follows that $a(X,Y) = v(t_1)$; $a(Y,Z) = v(t_2)$; $a(X,Z) = v(t_1+t_2)$. From (2.30), because of decomposability, we obtain the following *characterization of decomposable and uniform discount laws:*

$$v(t_1)\,v(t_2) = v(t_1+t_2);\ \forall(t_1 \geq 0,\ t_2 \geq 0) \tag{2.50'}$$

It is known that in the hypothesis that is valid for $u(t)$ and $v(t)$, the functional equations (2.50) and (2.50') are satisfied only by exponential functions; this proves the theorem[24].

If $\approx$ is uniform and strongly decomposable, and then symmetric, in (2.49) this results in $\delta = \theta$. The exchange factors then assume the form

$$g(t) = e^{\delta t},\ \forall t \in \Re \tag{2.51}$$

which satisfies (2.40). Equation (2.51), which is a particular example of (2.36), summarizes the exponential regime in the symmetric hypothesis and for all choices of δ identifies an exchange law that is strongly decomposable and uniform. Briefly, *the exchange exponential laws, and only those laws, correspond to indifferent relations that are equivalences that are uniform in time*[25].

24 The previous result can be deduced directly by observing that intensity depends on the initial time X and on the current time T, but if the law is decomposable the intensity must depend at the most on T, $\forall X$, while if the law is uniform the intensity must depend at the most on T-X. Then if the law is decomposable and uniform, both principles $\forall X$ being valid, it is necessary and sufficient that the intensity does not depend on any time variables, and it is constant; for compound accumulation laws, which in the continuous case lead to the exponential laws, see Chapter 3.

25 In the strongly decomposable and uniform law, which follows from a relation of uniform equivalence, the curves $S = \varphi(T)$, which correspond to the equivalence classes that are characterized, because of uniformity, by the further property of *invariance by translation*. Therefore, they follow by only one curve, which is translated continuously with a movement rigid and parallel to the time axes. The exponential curves $S = k\,e^{\delta T} = e^{\delta(T-T')}$ (where $T' = ln\ k/\delta$) are obtained, such that all the supplies equivalent to (T_0,S_0), so that all the supplies and only them, are represented by a point on the curve obtained putting $k = S_0\ e^{\delta T_0}$.